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FARRELL COHOMOLOGY OF GL(n, Z)

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ABSTRACT
The Tate–Farrell cohomology of GL(n, Z) with coefficients in Z/p is computed for p an odd prime and $p - 1 \leqq n \leqq 2p - 3$. Its size depends on the Galois structure of the class group of the cyclotomic field Q($\sqrt[p]{1}$) and is shown to be quite large in general.

Let p be an odd prime number. The mod p cohomology of GL(n, Z) and its subgroups of finite index interests us for several reasons. It is closely connected with the K-theory of Z. According to work of Dwyer and Friedlander [DF], the stable mod p cohomology of GL(n, Z) should be predictable from the Lichtenbaum conjecture. Hence an independent determination of the low dimensional cohomology of GL(n, Z) with large n could test the conjecture.

A second reason is connected with Serre's conjecture [Se] about representations of the Galois group of the algebraic closure of Q into GL(2, Z/p). These are supposed to be attached to modular forms and determined by the action of the Hecke algebra mod p. The Hecke eigenvalues of classical modular forms of weight 2 can be determined from the mod p cohomology of congruence subgroups of GL(2, Z). We would like to study the natural generalization of this conjecture to larger n. The interesting dimensional cohomology here is in the "cuspdal range" clustered around $n(n+1)/4$. Another good question is whether p-torsion in the integral cohomology will yield nontrivial Galois representations.

The link between the cohomology of GL(n, Z) and the topology of the set of lattices in n-dimensional euclidean space $\mathbf{E}^n$ gives a third reason. Using this link, Soule [So] computed completely the cohomology of GL(3, Z). For big n,

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it is unlikely that either the cohomology or the lattices will ever be completely understood, but they should shed partial light on each other, as for instance in [A1]. The mod p cohomology in particular will be connected with the subset of lattices that possess an automorphism of order p.

Unfortunately, it seems to be very hard to come to grips with the mod p cohomology of $\mathrm{GL}(n, \mathbf{Z})$ in general. As a preliminary approach to the problem, in this paper we will determine the Farrell cohomology of $\mathrm{GL}(n, \mathbf{Z})$ with trivial coefficient module $\mathbf{Z}/p$ for $n < 2p - 2$. (The same techniques could be used with any coefficient module.) The Farrell cohomology HF* is a generalization of Tate cohomology to groups of finite virtual cohomological dimension (vcd). The Farrell and ordinary cohomologies of $\mathrm{GL}(n, \mathbf{Z})$ coincide above the vcd $v = n(n-1)/2$. Below v, there is a map from H* to HF* which fits into a long exact sequence whose third term is the homology of $\mathrm{GL}(n, \mathbf{Z})$ with coefficients in the Steinberg module. Mark McConnell and I (work in progress) have shown that in positive dimensions below the vcd, roughly half of the Farrell cohomology lifts to the ordinary cohomology, a surprising result in view of the large size of the former, as seen in Theorem A below.

Using a spectral sequence due to K. Brown [B], the Farrell cohomology is completely determined by the normalizers of cyclic subgroups of order p in $\mathrm{GL}(n, \mathbf{Z})$. We find these normalizers in the first section, and derive Theorem A in the next section.

THEOREM A. *Let p be an odd prime, $p - 1 \leqq n \leqq 2p - 3$, $m = n - p + 1$. Set $\gamma_1 = \mathrm{GL}(m, \mathbf{Z})$ and $\gamma_2 = \{M \in \gamma_1 \mid$ first row of $M \equiv (*, 0, \ldots, 0) \bmod p\}$. Let $\mathbf{F}$ be the finite field with p elements, K the cyclotomic extension of $\mathbf{Q}$ generated by a primitive p-th root of unity, and Cl the ideal class group of K. For any $x \in \mathrm{Cl}$, let $s(x)$ denote the order of the stabilizer of x in $\Delta = \mathrm{Gal}(K/\mathbf{Q})$. Let $W(x, b, c)$ denote an $\mathbf{F}$-vector space whose dimension is the number of subsets I with c element of $\{2, 4, \ldots, (p-3)\}$ such that $s(x)$ divides ($\Sigma_I i$ + the greatest integer in $(b+1)/2$).*

Denote Farrell cohomology by HF. For any $t \in \mathbf{Z}$, choose $T \in \mathbf{Z}$ such that $T \equiv t \bmod (2p-2)$ and $T > n(n-1)/2 = \mathrm{vcd}(\mathrm{GL}(n, \mathbf{Z}))$. Then for all $t \in \mathbf{Z}$,*

$$\mathrm{HF}^*(\mathrm{GL}(n, \mathbf{Z}), \mathbf{F}) = \bigoplus_{x \in \mathrm{Cl}/\Delta} \bigoplus_{j=1,2} \bigoplus_{b+c+d=T} H^d(\gamma_j, \mathbf{F}) \otimes W(x, b, c).$$

In the special case $n = p - 1$, $m = 0$, we understand $\gamma_1 = \{1\}$ and $H^d(\gamma_2, \mathbf{F}) = 0$ for all d.

One thing to notice about the Farrell cohomology is the dependence on the

Galois structure of the ideal class group of the cyclotomic field K. The more classes and the smaller their stabilizers, the larger the Farrell cohomology. For large p, this means we may expect many unusual classes also in the ordinary cohomology of $\mathrm{Gl}(n, \mathbf{Z})$.

A second point of interest is that the ordinary cohomology (below the vcd) of Hecke-type congruence subgroups of $\mathrm{GL}(n - p + 1, \mathbf{Z})$ of level p begins to appear when $n > p$. Any alternative computation of the Farrell cohomology would then give information about this ordinary cohomology as well. In the work of McConnell and myself mentioned above, we are developing such alternatives in terms of the geometry of the space of lattices in $\mathbf{E}^n$.

In the last section we make some easy comparisons with the ordinary mod p cohomology of $\mathrm{GL}(n, \mathbf{Z})$ and its torsion-free congruence subgroups.

I wish to thank W. Sinnott for information about the Galois action on ideal class groups and R. Gold for providing the proof of Lemma 3.

1. Subgroups of order p in $\mathrm{GL}(n, \mathbf{Z})$

Let ζ be a primitive p-th root of unity, fixed once for all. Set $K = \mathbf{Q}(\zeta)$ and U = the units in $\mathbf{Z}[\zeta]$. If A is a fractional ideal in K, $[A]$ shall denote its ideal class. Let Δ be the Galois group of $K/\mathbf{Q}$ and μ the group of order p generated by ζ.

Our first task is to describe all $\mathbf{Z}\mu$-modules up to isomorphism. First we have the trivial module $\mathbf{Z}$ and any fractional ideal A of K. We can also form an indecomposable module M out of $\mathbf{Z}$, A and an arbitrary element $a \in A$ as follows. Let $\mathbf{Z}z$ be a free $\mathbf{Z}$-module and set $M = A \oplus \mathbf{Z}z$ as abelian group. μ acts on A in the usual way, and we set $\zeta z = a + z$. We denote the module M by the symbol (A, a). The following theorem and its proof may be found in [CR, p. 508].

THEOREM 1. *Any $\mathbf{Z}\mu$-module M which is free and finitely generated as a $\mathbf{Z}$-module is isomorphic to a direct sum*

$$(A_1, a_1) \oplus \cdots \oplus (A_r, a_r) \oplus A_{r+1} \oplus \cdots \oplus A_k \oplus Y$$

where the A_i are fractional ideals in K, $a_i \in A_i$, $a_i \notin (\zeta - 1)A_i$ and Y is a trivial $\mathbf{Z}\mu$-module. The isomorphism class of M is determined by the integers r, k, and the $\mathbf{Z}$-rank of Y, and by the ideal class of the product $A_1 \cdots A_k$.

Now suppose the $\mathbf{Z}$-rank of M to be n, where n shall be fixed and less than $2p - 2$. It follows that the only possibilities for M are of the form Y, $A \oplus Y$ and

$(A, a) \oplus Y$. In the last case we may and will choose a in $\mathbf{Q}$. From now on we assume that M is faithful, which eliminates the first possibility. Choosing a $\mathbf{Z}$-basis for M we obtain an embedding $\rho : \mu \to \mathrm{GL}(n, \mathbf{Z})$ and we wish to determine the normalizer of the image of ρ. Note that $\operatorname{Im} \rho_1$ is conjugate to $\operatorname{Im} \rho_2$ if and only if for some σ in Δ, $\rho_1 \circ \sigma$ is isomorphic to ρ_2. This is the reason that the first direct sum in Theorem A is over the Δ-orbits of Cl rather than Cl itself.

For any $m \geqq 1$, set $\gamma(m) = \mathrm{GL}(m, \mathbf{Z})$. We adopt the notation $\gamma(m, p)$ for the subgroup of $\gamma(m)$ whose first row is congruent to $(*, 0, \ldots, 0)$ modulo p. We let λ stand for the homomorphism $\gamma(m, p) \to (\mathbf{Z}/p)^{\times}$ sending such an element to $*$ modulo p. By convention, we set $\gamma(0) = \gamma(0, p) = \{1\}$.

THEOREM 2. *Let the image of ρ have normalizer N and centralizer C. Denote by S the stabilizer of the ideal class $[A]$ in the Galois group Δ. Then there is an exact sequence*

$$1 \to C \to N \to S \to 1.$$

To describe this further, we distinguish two cases: (1) $M \approx A \oplus Y$ and (2) $M \approx (A, a) \oplus Y$. Set $m = n - p + 1$. In Case 1 we set $\gamma = \gamma(m)$ and, in Case 2, $\gamma = \gamma(m, p)$. In either case let N_0 be the semidirect product of $U \times \gamma$ and S, where S acts trivially on γ and via the Galois action on U. Thus, the group law in N_0 is given by the formula $(\xi, \delta, \sigma)(\xi', \delta', \tau) = (\xi\sigma(\xi'), \delta\delta', \sigma\tau)$. Then N is a subgroup of N_0 and the map from N to S is induced by the obvious projection. Finally, we have

Case 1: $N \approx N_0$.

Case 2: $N \approx \{(\xi, \delta, \sigma) \in N_0 \mid \xi \equiv \lambda(\delta) s \bmod (\zeta - 1)\}$, where σ and s are related by the formula $\sigma(\zeta) = \zeta^s$ for $s \in (\mathbf{Z}/p)^{\times}$.

We remark that γ is embedded into the automorphism group of M in a natural way: in Case 1 as the automorphism group of Y and in Case 2 as the automorphism group of the $\mathbf{Z}$-span of z and Y.

We begin the proof of the theorem by observing that there is a natural isomorphism from N to the group $\{\phi \in \mathrm{Isom}_{\mathbf{Z}}(M, M) \mid \phi\zeta\phi^{-1} = \sigma(\zeta)$ for some $\sigma \in \Delta\}$ which maps C to the subgroup of ϕ for which $\sigma = 1$. We identify N and C with these sets for the rest of the proof. We have an obvious exact sequence $1 \to C \to N \to \Delta$. The following lemma will determine the image of the last arrow.

LEMMA 3. *Suppose $M = A$ and $\phi \in N$ with $\phi\zeta\phi^{-1} = \sigma(\zeta)$. Then $[A] = [\sigma A]$*

in the ideal class group. Conversely if $[A]=[\sigma A]$ *for some* $\sigma\in\Delta$, *there exists* $\phi\in N$ *such that* $\phi\zeta\phi^{-1}=\sigma(\zeta)$.

PROOF (R. Gold). Let ϕ be given. Then $\phi\zeta\phi^{-1}=$ multiplication by $\sigma(\zeta)$ on A. On the other hand, viewing $\sigma\in\mathrm{Isom}_{\mathbf{Z}}(A,\sigma A)$, we have $\sigma\zeta\sigma^{-1}=$ multiplication by $\sigma(\zeta)$ on $\sigma(A)$. But $\mathbf{Q}A=\mathbf{Q}\sigma A=K$. Then $\phi\zeta\phi^{-1}=\sigma\zeta\sigma^{-1}\in\mathrm{Isom}_{\mathbf{Q}}(K,K)$ so that $\sigma^{-1}\phi$ is a $\mathbf{Q}$-isomorphism of K which commutes with ζ, hence with any polynomial in ζ, i.e. with anything in K. Thus $\sigma^{-1}\phi$ must be multiplication by some element $x\in K$. So $xA=\sigma^{-1}\phi A=\sigma^{-1}A$ which implies that $[A]=[\sigma A]$.

Conversely, if $[A]=[\sigma A]$, there exists $x\in K$ such that $xA=\sigma^{-1}A$. Then $\phi=\sigma\circ$(mult. by x) is easily seen to lie in N and $\phi\zeta\phi^{-1}=\sigma(\zeta)$.

LEMMA 4. *Suppose* $M=A$ *and* $\phi\in C$. *Then* ϕ *is given by multiplication by some element of* U.

PROOF. In the notation of the proof of Lemma 3, $\sigma=1$ and ϕ must be multiplication by some $x\in K$. Since $xA=A$, we must have $x\in U$.

To continue the proof of the theorem, we introduce the notation M^{μ} for the fixed points of μ in M, v for the sum of the elements of μ in the group ring $\mathbf{Z}$, and $M[v]$ for the kernel of v acting on M. Note that $\phi\in N$ implies that $\phi\mu\phi^{-1}=\mu$ so that ϕ preserves both M^{μ} and $M[v]$.

In Case 1, $M^{\mu}=Y$ and $M[v]=A$. Hence any $\phi\in N$ preserves both A and Y. The statement of the theorem in this case now follows from Lemmas 3 and 4.

Case 2 is considerably more complicated. Again we have $M[v]=A$. The following lemma determines M^{μ}.

LEMMA 5. *The* μ*-invariants in* (A,a) *are rank* 1, *spanned by* $b+pz$, *where* $b=-p(\zeta-1)^{-1}a$.

PROOF. If $x\in A$ and $t\in\mathbf{Z}$, compute $\zeta(x+tz)=\zeta x+t(a+z)=(\zeta x+ta)+tz$. So $x+tz\in M^{\mu}$ iff $\zeta x+ta=x$ iff $ta=-(\zeta-1)x$. Since $a\notin(\zeta-1)A$, we must have p dividing t. Then x is a multiple of b and t is the same multiple of z.

Now any $\phi\in N$ preserves $M[v]=A$ and $M^{\mu}=\langle b+pz\rangle\oplus Y$. Let $R=\langle z\rangle\oplus Y$, so that $M=A\oplus R$ as abelian groups. We can write $\phi=\alpha+\beta+\delta$, where $\alpha\in\mathrm{Isom}_{\mathbf{Z}}(A,A)$, $\beta\in\mathrm{Hom}_{\mathbf{Z}}(R,A)$, and $\delta\in\mathrm{Isom}_{\mathbf{Z}}(R,R)$ with $\delta Y\subset p\langle z\rangle\oplus Y$. Then $\phi^{-1}=\alpha^{-1}+(-\alpha^{-1}\beta\delta^{-1})+\delta^{-1}$.

Suppose $\phi\zeta\phi^{-1}=\sigma(\zeta)=\zeta^{S}$. Then on A we have $\alpha\zeta\alpha^{-1}=\sigma(\zeta)$, so that by

Case 1 of the theorem (with $Y = 0$) we get for $x \in A$ that $\alpha(x) = \xi\sigma(x)$ for some $\xi \in U$. Then $\alpha^{-1}(x) = \sigma^{-1}(\xi^{-1}x)$.

Now suppose $r \in R$, $r = cz + y$ and $\delta^{-1}(r) = dz + y^*$ where $c, d \in \mathbf{Z}$. Then

$$\phi\zeta\phi^{-1}(r) = \phi\zeta[-\alpha^{-1}\beta\delta^{-1}(r) + \delta^{-1}(r)]$$
$$= \phi[-\zeta\alpha^{-1}\beta\delta^{-1}(r) + da + \delta^{-1}(r)] = (1-\zeta^s)\beta\delta^{-1}(r) + d\alpha(a) + r.$$

On the other hand, this is supposed to equal the result of ζ applied s times to r, namely, $cu_\sigma a + r$, where u_σ denotes the circular unit $(1-\zeta^s)/(1-\zeta) = 1 + \cdots + \zeta^{s-1}$. Since $a \in Q$, we obtain that $(1-\zeta^s)\beta\delta^{-1}(r) + d\xi a = cu_\sigma a$. Since δ is an isomorphism, this shows that β will be determined uniquely by α, δ and σ. Conversely, given α, δ and σ we can define β and hence ϕ satisfying the previous equations, if and only if $1-\zeta^s$ divides $cu_\sigma - d\xi$ for every choice of r (since $1-\zeta^s$ does not divide a). If this is so for r's of the form $z + y$ it will be so for all r. So we shall set $c = 1$.

Our condition for the existence of ϕ given α, δ and σ is that $s \equiv u_\sigma \equiv d\xi \bmod(\zeta - 1)$. Extending $\{z\}$ to a $\mathbf{Z}$-basis of R, we can express δ as a matrix in $\gamma(m, p)$ so that $d^{-1} \equiv \delta_{11} = \lambda(\delta)$. Our condition then becomes $\lambda(\delta)s \equiv \xi$ as claimed.

2. Farrell cohomology

Conjugacy classes of subgroups of order p in $\Gamma = \mathrm{GL}(n, \mathbf{Z})$ are in one-to-one correspondence with isomorphism classes of $\mathbf{Z}\mu$-modules which are free of rank n as $\mathbf{Z}$-modules. Assume $n < 2p - 2$ so that every elementary abelian p-subgroup of Γ has rank 0 or 1. Denote the p-part of the Farrell cohomology of Γ with coefficients in the $\mathbf{Z}\Gamma$-module E by $\mathrm{HF}^*(\Gamma, E)_{(p)}$. Then Theorem 6.7 and Corollary 7.4 in Chapter X of [B], imply the following:

THEOREM 3. *Recall that $n < 2p - 2$. Then*

(1) *Γ and each of its subgroups has p-periodic Farrell cohomology*;

(2) $\mathrm{HF}^*(\Gamma, E)_{(p)} \approx \bigoplus \mathrm{HF}^*(N(P), E)_{(p)}$, *where the direct sum runs over a set of representatives $\{P\}$ for the conjugacy classes of subgroups of order p in Γ and $N(P)$ denotes the normalizer of P in Γ.*

If $n < p - 1$, there are no P's and $\mathrm{HF}^*(\Gamma, E)_{(p)}$ vanishes. Hence we will assume from now on that $p - 1 \leqq n \leqq 2p - 3$. From the results in Section 1 we know that $\{P\}$ can be indexed by a set of representatives $\{A\}$ of the ideal class group of $K = \mathbf{Q}(\zeta)$. For each A, define $N(A)$ (resp. $N'(A)$) to be the normalizer N appearing in Case 1 (resp. Case 2) of Theorem 2 of Section 1. If

$n = p - 1$, we do not define $N'(A)$ and omit terms referring to it in what follows. We then have immediately:

COROLLARY 4. $\mathrm{HF}^*(\Gamma, E)_{(p)} \approx \bigoplus [\mathrm{HF}^*(N(A), E)_{(p)} \oplus \mathrm{HF}^*(N'(A), E)_{(p)}]$.

We must now evaluate the terms on the right hand side. Since the Farrell cohomology is p-periodic, we may without loss of generality assume that * exceeds the virtual cohomological dimension of Γ. Then the Farrell cohomology equals the ordinary cohomology and we may drop the letter F as long as this assumption about * is in force.

Fix A and let N be $N(A)$ or $N'(A)$. Apply the Hochschild–Serre spectral sequence to the exact sequence in Theorem 2, to obtain

$$E_2^{s,t} = H^s(S, H^t(C, E))_{(p)} \Rightarrow H^{s+t}(N, E)_{(p)}.$$

Since the order of S is prime to p, this reduces to $H^0(S, H^t(C, E)_{(p)}) = H^t(N, E)_{(p)}$. Since we know C so explicitly, in particular cases we should be able to compute the left hand side.

There are two obstacles to going on at this point. First, we don't know much about the cohomology of γ, even with trivial coefficients, except for some weak lower bounds. Second, we don't completely know the structure of U as $\mathbf{Z}\Delta$-module.

To obtain more explicit results, we assume that E is the finite field $\mathbf{F}$ with p elements, viewed as a trivial module for Δ. We can then drop the (p)-subscripts. We have an exact sequence of Δ-modules $1 \to \mu \to U \to U' \to 1$, where $U' \approx \mathbf{Z}^{(p-3)/2} \times \mathbf{Z}/2$. As a sequence of $\mathbf{Z}$-modules it splits, so we choose a splitting and view U' as a subgroup of U. We then have an exact sequence of Δ-modules $1 \to \mu \to C \to C' \to 1$, where C' is a p-torsionfree subgroup of C such that C is the direct product $\mu \times C'$ as groups, although not as Δ-modules. In Case 1, $C' = U' \times \gamma$. In Case 2, the fact that $\zeta \equiv 1 \bmod (1 - \zeta)$ allows us to set

$$C' = \{(\xi, \delta, 1) \in N_0 \mid \xi \in U' \text{ and } \xi \equiv \lambda(\delta) \bmod (\zeta - 1)\}.$$

In any case, the Hochschild–Serre spectral sequence degenerates and we get the isomorphism of S-modules:

$$H^t(C, \mathbf{F}) \approx \bigoplus_{a+b=t} H^a(C', H^b(\mu, \mathbf{F})).$$

For any integer m, let us write m' for the largest integer contained in $(m + 1)/2$. Let $\mathbf{F}[m]$ denote $\mathbf{F}$ viewed as Δ-module where $\sigma \in \Delta$ acts by the formula $\sigma \cdot x = s^m x$ when $\sigma(\zeta) = \zeta^s$. Then it is easy to see that $H^b(\mu, \mathbf{F}) \approx \mathbf{F}[b']$ as Δ-modules.

Now use the Kunneth formula to compute the cohomology of C'. In Case 1 C' is $U' \times \gamma$. In Case 2 the index of C' in $U' \times \gamma$ is prime to p and a set of representatives of the quotient can be found in $U' \times \{1\}$. Since U' is abelian, from the Hochschild–Serre spectral sequence we get $H^a(C', \mathbf{F}) = H^a(U' \times \gamma, \mathbf{F})$. Thus in both cases we have

$$H^a(C', \mathbf{F}) \approx \bigoplus_{c+d=a} H^c(U', \mathbf{F}) \otimes H^d(\gamma, \mathbf{F}).$$

Now Δ acts trivially on the second tensor factor. As for the first, write $\mathrm{Hom}(U', \mathbf{F}) = V$. Since the cohomology of U' mod p is that of a torus of rank $(p-3)/2$, we have $H^c(U', \mathbf{F}) \approx \Lambda^c(V)$ as Δ-module. From Propositions 8.10 and 8.13 of chapter 8 in [W], we have that $V \approx \bigoplus_{e=2,4,6,\ldots,(p-3)} \mathbf{F}[e]$.

Putting this all together we have:

$$H^t(C, \mathbf{F}) \approx \bigoplus_{a+b=t} H^a(C', \mathbf{F}[b']) \approx \bigoplus_{a+b=t} \bigoplus_{c+d=a} H^c(U', \mathbf{F}) \otimes H^d(\gamma, \mathbf{F}) \otimes \mathbf{F}[b']$$

$$\approx \bigoplus_{b+c+d=t} \Lambda^c\left(\bigoplus_{e=2,4,6,\ldots,(p-3)} \mathbf{F}[e]\right) \otimes \mathbf{F}[b'] \otimes H^d(\gamma, \mathbf{F}).$$

Now the S-invariants in this give $H^t(N, \mathbf{F})$. Note that S acts trivially on $H^d(\gamma, \mathbf{F})$. Letting $\#S$ denote the number of elements in S, we see easily that S acts trivially on $\mathbf{F}[m]$ if and only if $\#S$ divides m. Therefore we have proved:

THEOREM 5. *With notation as above, let $W(b, c)$ denote an $\mathbf{F}$-vector space whose dimension is the number of subsets I with c elements of $\{2, 4, \ldots, (p-3)\}$ such that $\#S$ divides $b' + \Sigma_I i$. Then*

$$H^t(N, \mathbf{F}) \approx \bigoplus_{b+c+d=t} H^d(\gamma, \mathbf{F}) \otimes W(b, c).$$

As a supplement, we prove:

LEMMA 6. *The right hand side in Theorem 5 is eventually periodic in t of period $2(p-1)$.*

PROOF. Let $r = (p-1)/2$. Suppose $t > \mathrm{vcd}(\gamma) + r$. Since γ has no p-torsion, $H^d(\gamma, \mathbf{F})$ vanishes for $d > \mathrm{vcd}(\gamma)$. Obviously $W(b, c)$ vanishes if $c > r - 1$. Hence the right hand side may be written

$$\bigoplus_{c=0,\ldots,r-1}\ \bigoplus_{d=0,\ldots,\mathrm{vcd}(\gamma)} H^d(\gamma, \mathbf{F}) \otimes W(t-c-d, c).$$

Now $W(b,c) = W(b+4r, c)$, since $(b+4r)' = b' + 2r$, and $\#S$ divides $2r$.

We remark that we could strengthen the lemma to give the period at twice $\#S$. However, we wanted to state a result independent of the ideal class $[A]$. If we take this periodicity into account for each choice of A, and apply this to Corollary 4, we obtain the full determination of $\mathrm{HF}^*(\Gamma, \mathbf{F})$. This gives Theorem A of the introduction.

The result simplifies if $n = p-1$ or p, for then $\gamma = \{1\}$ or $\{\pm 1\}$. A very simple summand then occurs for an ideal class A whose Galois stabilizer S is trivial. In such a case, $W(b,c)$ = number of subsets of c elements in I independently of b, and $H^d(\gamma, \mathbf{F}) = 0$ except when $d = 0$, so that as long as $t \gg 0$, $H^t(N, \mathbf{F})$ is a vector space of dimension $2^{(p-3)/2}$, independently of t. So this summand has period 1 in t. Thus we have the simple

COROLLARY 7. *Let $n = p-1$ or p and suppose $K = \mathbf{Q}(\zeta_p)$ has an ideal class fixed only by the identity in* Gal($K/\mathbf{Q}$). *Then for any $t \in \mathbf{Z}$,* $\dim HF^t(\Gamma, \mathbf{F}) \geqq 2^{(p-2)/3}$.

The existence of a class with trivial stabilizer seems to be a common property of primes. In fact, for any field F let $H(F)$ denote its class group and $h(F)$ the order of $H(F)$. Let c denote complex conjugation, and $K = \mathbf{Q}(\zeta_p)$ as usual.

LEMMA 8. *Let $x \in H(K)$ of prime order l be such that $c(x) = -x$. Assume that l does not divide $h(F)$ nor $[K:F]$ for any maximal proper nonreal subfield F of K. Then the stabilizer S of x in* Gal($K/\mathbf{Q}$) *is trivial.*

PROOF. Suppose not. Let $T \subset \mathrm{Gal}(K/\mathbf{Q})$ be cyclic of prime order t stabilizing x. Set $F = K^T$, so that $[K:F] = t$. Since c is not in T, F is a nonreal field. By hypothesis $l \neq t$. We have that $\mathrm{conorm}_{K/F}(\mathrm{norm}_{K/F}(x)) = \Pi_{\sigma \in T} x^\sigma = tx$. But $\mathrm{norm}_{K/F}(x)$ has order dividing l in $H(F)$, so, by hypothesis, it must be 0. Thus tx and lx both equal 0, so that $x = 0$. Contradiction.

If we now look at Table II, pp. 150 ff. in [H], we see that for every prime p from 37 to 97, K has classes whose stabilizers in Δ are trivial. For instance, if $p = 41$, use $l = 11$, etc. However, I don't know how to prove this to be the case for an infinite number of p.

3. Comparison with ordinary cohomology

Let St denote the Steinberg module for $\Gamma = GL(n, \mathbf{Z})$ tensored with $\mathbf{F}$, the finite field with p elements. Let $N = \mathrm{vcd}(\Gamma) = n(n-1)/2$. From p. 280 of [B] we have the long exact sequence

$$\cdots \to H_{N-i}(\Gamma, \mathrm{St}) \to H^i(\Gamma, \mathbf{F}) \to HF^i(\Gamma, \mathbf{F}) \to H_{N-i-1}(\Gamma, \mathrm{St}) \to \cdots.$$

Thus immediately $H^i(\Gamma, \mathbf{F}) \to HF^i(\Gamma, \mathbf{F})$ is an isomorphism if $i > N$ and a surjection if $i = N$. We can show that it is also an isomorphism if $i = N$ and a surjection if $i = N - 1$. In fact, we can realize the Steinberg module using modular symbols as in [AR]. Using Theorem 4.1 in that paper, we easily see that $H_0(\Gamma, \mathrm{St}) = 0$. We hope to study further $H_j(\Gamma, \mathrm{St})$ for $j > 0$ in a future paper.

Now let Γ' be a torsionfree normal subgroup of Γ of finite index. The Hochschild–Serre spectral sequence yields

$$E_2^{a,b} = H^a(\Gamma/\Gamma', H^b(\Gamma', \mathbf{F})) \Rightarrow H^t(\Gamma, \mathbf{F}).$$

Now we use the fact that $\dim H^t(\Gamma, \mathbf{F}) \geqq \dim \mathrm{HF}^t(\Gamma, \mathbf{F})$ for $t = N-1, N$. We also note that $H^0(\Gamma/\Gamma', H^N(\Gamma', \mathbf{F})) = 0$ by using modular symbols again and Borel–Serre duality as in [A]. We see immediately that $\bigoplus_{b=0,\ldots,N-1} H^b(\Gamma', \mathbf{F})$ cannot vanish. In further work we hope to derive stronger nonvanishing results on $\bigoplus_{b=0,\ldots,N-1} H^b(\Gamma', \mathbf{F})$ as a Γ/Γ'-module using this spectral sequence.

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